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1995 J. Phys.: Condens. Matter 7 6079

(http://iopscience.iop.org/0953-8984/7/30/012)

J. Phys.: Condens. Matter 7 (1995) 6079–6096. Printed in the UK

The spin-1/2 Heisenberg spin ladder with bond alternation

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Received 14 March 1995

Abstract. The spin-1/2 Heisenberg spin ladder model is studied by via a perturbation calculation and via bosonization. We mainly focus on the case of ferromagnetic interchain coupling in relation to the Haldane problem. Using the GKO coset construction of the conformal field theory, we bosonize the system in terms of the SU(2) (spin) sector and a residual Ising sector. The interchain interaction generically drives the system to strong-coupling massive phases (the Haldane phase or the dimerized phase). However, it is suggested that the criticality appears for particular choices of the couplings.

1. Introduction

Recently, spin ladder models have attracted a considerable interest. When the interchain coupling is antiferromagnetic, they are considered to be models of the so-called high-T_c material [1–3] and they have been investigated by several authors [4–7]. Moreover, they are realized experimentally as $(VO)_2P_2O_7$ or $Sr_2Cu_4O_6$ [8, 9].

On the other hand, ladders with the ferromagnetic interchain coupling are interesting in relation to the Haldane systems [10, 11] (see [12] for a review) and have been extensively studied both numerically [13, 14] and analytically [15, 16]. That is, they smoothly interpolate between the fairly well established $S = 1/2$ chains and the $S = 1$ chain, which is far from completely settled, as the interchain coupling is varied.

In the present paper, we mainly treat the following Hamiltonian

$$\mathcal{H} = \sum_i (\boldsymbol{S}_{2i-1} \cdot \boldsymbol{S}_{2i} + \boldsymbol{T}_{2i-1} \cdot \boldsymbol{T}_{2i}) + J' \sum_j \left(\boldsymbol{S}_{2j} \cdot \boldsymbol{S}_{2j+1} + \boldsymbol{T}_{2j} \cdot \boldsymbol{T}_{2j+1}\right) + J_K \sum_i \boldsymbol{S}_i \cdot \boldsymbol{T}_i \tag{1}$$

where the alternating intrachain coupling J' and the interchain coupling J_K can vary from $-\infty$ to $+\infty$. The spin-half operators $\{\boldsymbol{S}_i\}$ and $\{\boldsymbol{T}_i\}$ independently form two $S = 1/2$ alternating Heisenberg chains. The present model includes several interesting models as special cases.

(1) The case $J_K \to -\infty$, J' = finite: the $S = 1$ alternating Heisenberg chain (strictly speaking, the Hamiltonian (1) reduces to the $S = 1$ chain sector and an infinitely massive sector).

(2) The case J_K = fixed, $J' \to -\infty$: the $S = 1$ Heisenberg spin ladder.

(3) The case $J_K \to -\infty$, $J' \to -\infty$: the $S = 2$ Heisenberg antiferromagnet.

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0953-8984/95/306079+18$19.50

Furthermore, analytic results are available for the following three limits.

(i) The case $J_K = 0$: the Hamiltonian reduces to two decoupled alternating Heisenberg chains. The exact Bethe *ansatz* solution [17] exists in the case without bond alternation ($J' = 1$). For generic values of J', the $S = 1/2$ alternating chain is investigated by several authors [18, 19].

(ii) The case $J' = 0$: the model becomes an assembly of the decoupled four-site clusters and it can be trivially solved.

(iii) The case $J_K \to +\infty$: we have a problem of decoupled dimers. The ground state is simply given by the product of singlet dimers.

It is well known [14] that the ferromagnetic limit ($-J_K \gg 1$) of the model (1) without bond alternation ($J' = 1$) has low-energy characteristics in common with the $S = 1$ Heisenberg model; a finite excitation gap and short-ranged correlations. On the other hand, the model is massless when $J_K = 0$ [20]. Therefore, at least one transition must occur in between. Up to now, however, the location of the critical point as well as its type has been controversial [13, 15, 16]. It is one of the main purposes of the present paper to determine them and clarify the origin of the Haldane phase. In the following, we also look for other critical points and obtain a qualitative picture of the phase diagram.

The present paper is organized as follows. In section 2, we perform the perturbation expansion around the two limiting cases: (i) the case $J_K \to +\infty$ (the dimer limit) and (ii) the case $J' = 0$. The expressions of the dispersion relation for the elementary excitation and the excitation gap are obtained up to the second order.

In order to investigate the neighborhood of the point ($J_K = 0$, $J' = 1$), we adopt the non-Abelian bosonization [21, 22] and the renormalization group calculation [23] in sections 3 and 4. These sections are the main part of the present paper.

In section 3, we map the two decoupled chains onto the continuum field theory. Then, we re-express it in terms of the Ising field theory and the Wess–Zumino–Witten (WZW) model [21, 24] using the well known coset construction [25, 26]. The resulting field theory is given by an interacting model of the Ising model and the level-2 SU(2) WZW model. The latter corresponds to the spin sector and is considered to be relevant for our analysis. This description is particularly suitable for investigating the case of the ferromagnetic J_K.

Then we treat the model by the renormalization group (RG) technique in section 4. To derive the 1-loop β-functions, the Kosterlitz-type renormalization scheme is used. Our result shows that the system flows to the strong-coupling regions in the infrared limit. Using the semiclassical treatment, we argue that the non-trivial strong-coupling fixed point exists for particular choices of the couplings (J_K, J'). Within this type of approximation, it belongs to the universality class of the level-1 (*not* level-2!) SU(2) WZW model. The relation to another type of coupled chain is also discussed briefly.

We summarize the results obtained in section 5.

2. Perturbative evaluation of excitation spectra

As was mentioned in section 1, our model (1) allows the perturbation expansion in the two limits: (i) $J_K \to \infty$ and (ii) $J' = 0$. In these limits, the ground state is given by a product of local singlet states. Hence a theorem concerning the excitation gap (Theorem 4.3 of [27]) can be applied in order to show that there exists a non-zero gap in some finite region around these limiting values of the coupling constants.

It is well known that the Lieb–Schultz–Mattis theorem [28, 29] can be extended straightforwardly [30] to the *XXZ* spin ladder without bond alternation. This tells us that if

the infinite-volume ground state of the ladder with an odd number of legs is not degenerate then the excitation is inevitably gapless. Unfortunately, the theorem gives no meaningful statement for the relevant cases, e.g. ladders with an even number of legs. Therefore, in order to obtain a qualitative picture of the excitation spectrum, we perform a perturbative calculation around the limits above mentioned. Theorem 4.3 of [27] guarantees that these expansions indeed have finite convergence radii.

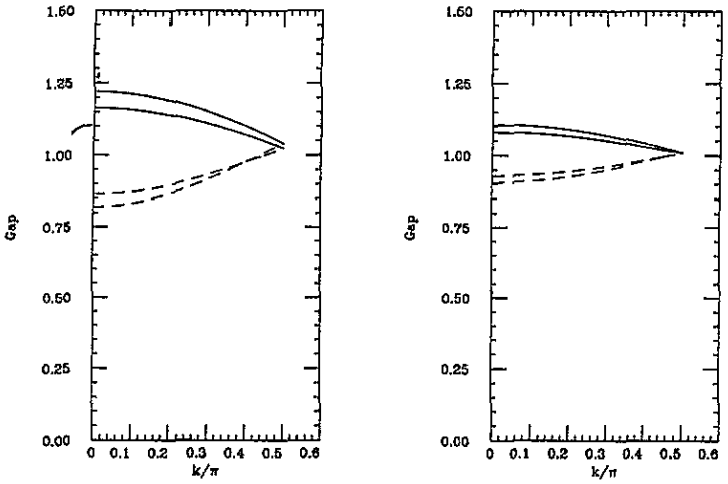


Figure 1. Dispersion relations $\omega_+(k)/J_K$ and $\omega_-(k)/J_K$ obtained by the strong-coupling expansion in section 2. (a) $J_K = 5$ and (b) $J_K = 10$. The solid and dashed lines denote ω_+ and ω_-, respectively.

First, we consider the type-(i) limit and expand the spectrum in $1/J_K$ (hereafter, we call it *the strong-coupling expansion*). In this limit, the ground state is a trivial dimer singlet state with singlet valence bonds sitting on every rung. Obviously, the first excited states are obtained by replacing one of the valence bonds by a triplet bond. For a homogeneous case $J' = 1$, the dispersion relation of this type of excitation is calculated using the perturbation expansion [4, 31] and a kind of mean-field calculation [5]. Since the calculation is rather straightforward, we give only the result up to the second order in $1/J_K$:

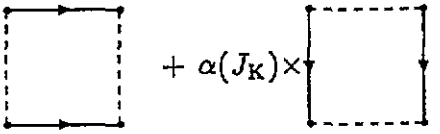
$$\omega_\pm(k) = J_K \left\{ 1 \pm \frac{1}{2J_K}(1 + J')\cos k + \frac{1}{8J_K^2}\left[3 + 3(J')^2 - 2J'\cos 2k\right] \right\} \tag{2}$$

where $\omega_+(k)$ ($\omega_-(k)$) denotes the upper (lower) branch of the spectrum and the wave number k runs half the Brillouin zone. We plot the dispersion $\omega_\pm(k)/J_K$ in figure 1 for two values of J_K. Setting $J' = 1$, we reproduce the known result [4, 31], as is expected. Using equation (2), the gap is given by the following formula:

$$\Delta = J_K \left\{ 1 - \frac{1}{2J_K}(1 + J') + \frac{1}{8J_K^2}\left[3 + 3(J')^2 - 2J'\right] \right\}. \tag{3}$$

Another expansion for the energy spectrum is possible around $J' = 0$ (we call it *the weak-coupling expansion*). When $J' = 0$, the system reduces to an assembly of the decoupled four-site clusters. The ground state of each cluster is always given by the

following singlet state (*singlet-A*):

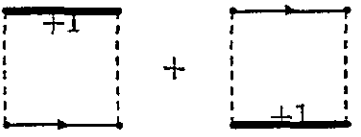


The arrows stand for the singlet valence bonds including their orientation and the quantity $\alpha(J_K)$ is defined by

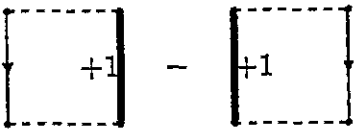
$$\alpha(J_K) = J_K - 1 + \sqrt{J_K^2 - J_K + 1}.$$

Note that $\alpha(J_K) \to -1/2$ as $J_K \to -\infty$. It interpolates between the two strong-coupling limits ($J_K \gg 1$ and $-J_K \gg 1$) smoothly.

The first excited state for $J_K > 0$ is different from that for $J_K < 0$. When the interchain coupling J_K is negative (i.e. ferromagnetic), the triplet state of the following type (*triplet A*)



is the first excited state of a single cell, whereas it is replaced by another triplet state (*triplet B*) for $J_K > 0$:



The thick lines denote triplet bonds which can take three values -1, 0, 1. Since a pair of $S = 1/2$ spins on each rung are symmetrized to form a single $S = 1$ spin in the $J_K \to -\infty$ limit, the excited state *triplet A* can be identified as a weak-coupling analogue of the excitation in the $S = 1$ dimer phase. On the other hand, the state *triplet B* is nothing but the elementary excitation considered in the strong-coupling expansion. The other sectors are always higher lying.

We performed the expansion up to the first order in J' to obtain the spectrum

$$\omega_>(k) = \alpha(J_K) - J_K + 1 - \frac{(1+\alpha(J_K))^2 J'}{2\left(1+\alpha(J_K)+\alpha^2(J_K)\right)} \cos k \tag{4}$$

for $J_K > 0$, and

$$\omega_<(k) = \alpha(J_K) + 1 - \frac{J'}{2\left(1+\alpha(J_K)+\alpha^2(J_K)\right)} \cos k \tag{5}$$

for $J_K < 0$. For $J' > 0$, the lowest excited state has momentum $k = 0$ and gives the gap

$$\Delta = \alpha(J_K) - J_K + 1 - \frac{(1+\alpha(J_K))^2 J'}{2\left(1+\alpha(J_K)+\alpha^2(J_K)\right)} \tag{6}$$

for $J_K > 0$, and

$$\Delta = \alpha(J_K) + 1 - \frac{J'}{2\left(1+\alpha(J_K)+\alpha^2(J_K)\right)} \tag{7}$$

for $J_K < 0$. The plot of it (figure 2) shows the peculiar behaviour of the gap in the interchain coupling J_K. That is, it is almost flat for the ferromagnetic side $J_K < 0$, while it grows almost linearly in J_K on the antiferromagnetic side. For $J_K < 0$, we also performed the weak-coupling expansion up to second order. Since the resulting expression is lengthy, we give it in appendix A. When $J_K \to -\infty$, the gap vanishes at $J' = 0.557$, which is close to the Haldane–dimer transition point $J' = 0.595 \pm 0.010$ of the $S = 1$ chain [32]. In figure 3, we also show the Haldane–dimer transition line $J' = J'_c(J_K)$ evaluated using the second-order result.

Expanding the right-hand side of (7) in $1/J_K$, we reproduce the strong-coupling result (3) up to the first order in J'. Comparing the gaps obtained in these two ways (figure 4), we can see that they are almost the same for $J_K \gtrsim 3$, although the regions of validity for the two expansions are quite different from each other. This implies that the crossover to the strong-coupling ($J_K \gg 1$) behaviour occurs rather fast. Therefore, we may conclude that the excitation gap is non-vanishing for $J_K \gg 1$ (actually, $J_K \gtrsim 3$) and for small J'.

There is an obvious equivalence mapping on the ground-state phase diagram of our Hamiltonian (1):

$$(J', J_K) \mapsto (1/J', J_K/J'). \tag{8}$$

Using this symmetry, our observation is extended to the region $J' \gg 1$. Thus the only remaining unsolved region is around $(J', J_K) = (1, 0)$. It does not allow the usual perturbation expansion. Hence, we use the mapping to the continuum field theory to analyse the model around this point in the next section.

3. Mapping to field theory

In the last section, we performed perturbation expansions from the two limits and found that there is always non-zero spin gap in the antiferromagnetic region $J_K > 0$ except for in the vicinity of the point $(J' = 1, J_K = 0)$. The region around this point (both ferromagnetic and antiferromagnetic) is not accessible by a simple perturbation. To complement the strong- or weak-coupling expansion, we adopt the bosonization technique based on the continuum field theory.

Our tactics is as follows. First, we map the two decoupled Heisenberg chains onto two independent field theoretical models, and then introduce interactions between them which correspond to the interchain coupling and the bond alternation. In doing this, we use the non-Abelian bosonization [21, 22] based on the Wess–Zumino–Witten (WZW) conformal field theory, instead of the usual Abelian bosonization. A detailed account of this method can be found in [33].

It is well known [34–36] that the antiferromagnetic $S = 1/2$ Heisenberg chain ($J' = 1$) reduces to the level-1 WZW model in the low-energy limit. This is also supported by the fact that the spinon S-matrix [37, 38] of the former coincides with the physical S-matrix [39] of the latter. In stead of a single free boson in the case of the Abelian bosonization, an SU(2) matrix field g is used in our formalism. Since we have two $S = 1/2$ chains at $J_K = 0$, we introduce two independent WZW fields g and $\tilde{g}$ correspondingly. The SU(2) matrix fields g and $\tilde{g}$ are governed by the following action [21, 24]:

$$S_{WZW}[g] = \frac{k}{8\pi}\int d^2x\ \mathrm{Tr}\left(\partial_\mu g\ \partial_\mu g^{-1}\right) + \frac{k}{12\pi}\int_B d^3x\ \epsilon_{\mu\nu\rho}\mathrm{Tr}\left(g^{-1}\partial_\mu g\ g^{-1}\partial_\nu g\ g^{-1}\partial_\rho g\right) \tag{9}$$

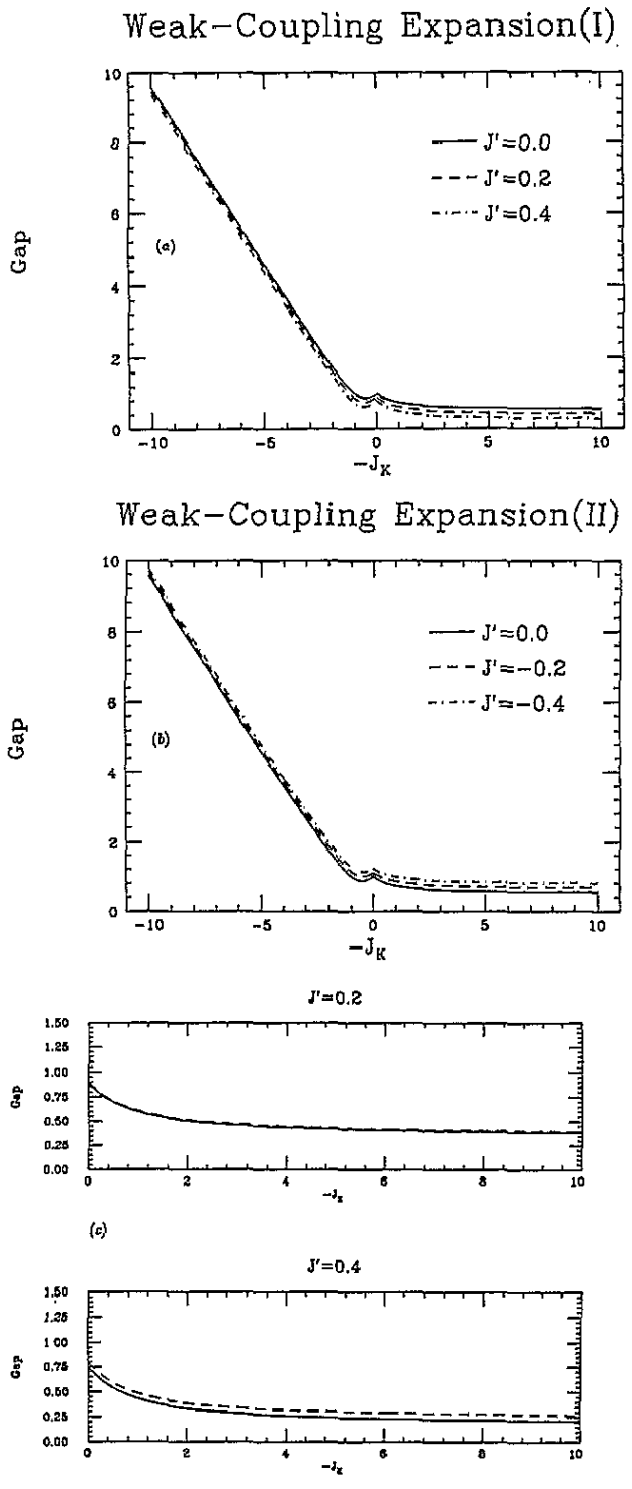


Figure 2. Dispersion relations $\omega_>$ and $\omega_<$ obtained by the weak-coupling expansion in J'. (a) $\omega(k)$ for $J' > 0$ and (b) for $J' < 0$. Comparisons between the first-order results (dashed lines) and the second-order ones (solid lines) are shown in figure 2(c).

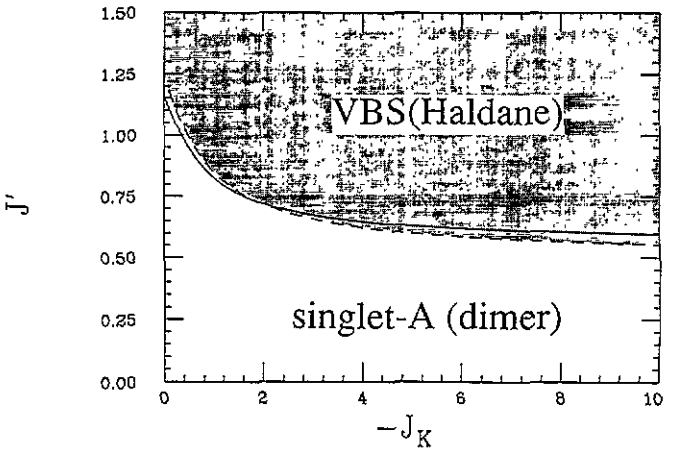


Figure 3. The transition line of the Haldane–dimer transition evaluated using the second-order result (solid line). The transition line determined by another method (a variational calculation [53]) is also plotted (dashed line). Note that an obvious symmetry mentioned in section 2 guarantees the existence of another transition line obtained by $J' \to 1/J'$, $J_K \to J_K/J'$.

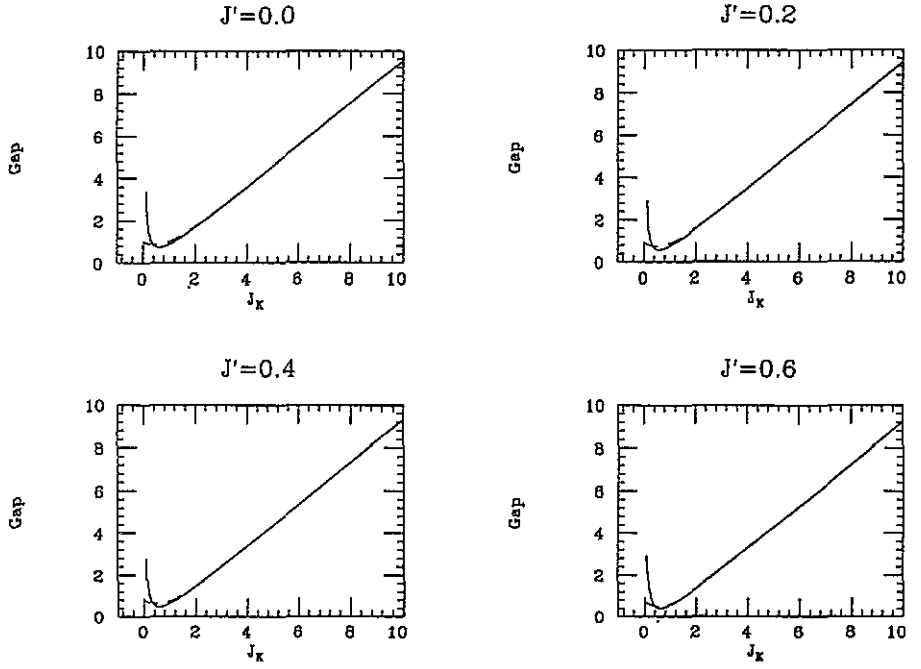


Figure 4. Comparison of the gap obtained in two ways: (i) the strong-coupling expansion (dashed lines) and (ii) the weak-coupling expansion (solid lines). Note that the two results are almost the same for $J_K \gtrsim 3$.

where the second integral is performed in the interior of the three-dimensional ball whose boundary is the two-dimensional space-time. The parameter k is called *level* and takes integral values. In our case, k is set equal to 1 or 2. For $k = 1$, the matrix field g can

be simply expressed in terms of a single free boson. For example, the matrix elements are given by [41]

$$g_{1/2}^{1/2} = g_{1/2,-1/2} = :e^{i\sqrt{2}\tilde{\phi}}: \qquad g_{1/2}^{-1/2} = -g_{1/2,1/2} = -:e^{i\sqrt{2}\phi}:.$$

The free boson field ϕ and its dual $\tilde{\phi}$ are defined by $\phi = (\phi_L + \phi_R)/2$ and $\tilde{\phi} = (\phi_L - \phi_R)/2$, where the chiral boson obeys $\langle \phi_L(z)\phi_L(w)\rangle = -\ln(z-w)$. Note that the field g cannot be realized by a single free boson for $k > 1$.

Using the fields g and $\tilde{g}$, the spin operator can be expressed as [22, 35]

$$\begin{aligned} S_i &\approx J_L + J_R + \text{constant} \times (-1)^i \operatorname{Tr}(g\sigma) \\ T_i &\approx \tilde{J}_L + \tilde{J}_R + \text{constant} \times (-1)^i \operatorname{Tr}(\tilde{g}\sigma) \end{aligned} \tag{10}$$

where the symbol σ denotes the Pauli matrices. In the above equations, J_L and J_R ($\tilde{J}_L$ and $\tilde{J}_R$) stand for the left and right SU(2) currents for the $S(T)$-chain, respectively, which satisfy the so-called level-1 Kac–Moody algebra (see [41] for a review). Hereafter, we use the tilde to denote the quantities of the T-chain. The low-energy effective Hamiltonian for the S-chain is simply expressed as the following normal-ordered current bilinears [22]:

$$\mathcal{H}_S = \frac{2\pi v_F}{L} \int_0^L dx \frac{1}{k+2} [:J_L \cdot J_L: + :J_R \cdot J_R:] \tag{11}$$

where the quantity $v_F = \pi/2$ corresponds to the 'light velocity' of the field theory. The Hamiltonian for the T-chain is obtained similarly after replacing $J_{R/L}$ by $\tilde{J}_{R/L}$. In this representation, the translational symmetry of the original lattice problem turns into a discrete symmetry $g \to -g$. In the following sections, we only consider the case where all parameters are set equal for the two chains. Hence all the expressions are symmetric under $g \to \tilde{g}$, $J \to \tilde{J}$.

Taking the operator-product expansions $S(i) \cdot S(i+1)$ and $T(i) \cdot T(i+1)$, we obtain the expression for the bond alternation

$$\operatorname{Tr} g \text{ for the } S\text{-chain} \qquad \operatorname{Tr} \tilde{g} \text{ for the } T\text{-chain.} \tag{12}$$

In terms of the two WZW models, the intrachain interaction can be written as

$$\lambda_1 \left(J_L \cdot J_R + \tilde{J}_L \cdot \tilde{J}_R\right). \tag{13}$$

The initial coupling $\lambda_1^{(0)}$ is positive, so it may be marginally irrelevant for $J_K = 0$. The interchain interaction and the bond alternation reduce to the following (Lorentz-invariant) interactions

$$\lambda_2 \left(J_L \cdot \tilde{J}_R + \tilde{J}_L \cdot J_R\right) \qquad \lambda_3 \left[\operatorname{Tr}(g\sigma) \cdot \operatorname{Tr}(\tilde{g}\sigma)\right] \tag{14}$$

and

$$\lambda_4 \left[\operatorname{Tr} g + \operatorname{Tr} \tilde{g}\right] \tag{15}$$

respectively. In our treatment, the spatial distance between S and T is negligible in the continuum limit. In order to preserve the renormalizability, we have to add another interaction

$$\lambda_5 \left[\operatorname{Tr} g \operatorname{Tr} \tilde{g}\right] \tag{16}$$

as a counter term. The initial values of the coupling constants are given by

$$\lambda_1^{(0)} < 0 \qquad \lambda_2^{(0)} = J_K \qquad \lambda_3^{(0)} \sim J_K \qquad \lambda_4^{(0)} = J' - 1 \qquad \lambda_5^{(0)} = 0.$$

To analyse the infrared behaviour of the present system, we derive the renormalization group β-function up to the 1-loop order. The calculation may be most easily done by

the renormalization procedure *à la* Kosterlitz (for an account of this method see [42] and references cited therein). A similar calculation was made by Strong and Millis [44] for a set of interactions without the alternation $\mathrm{Tr}\, g + \mathrm{Tr}\, \tilde{g}$. Hence we omit the calculational detail (see appendix B for a comment on the calculation) and give only the result up to the 1-loop order;

$$\begin{aligned}
\frac{\mathrm{d}\lambda_1}{\mathrm{d}\ln L} &= 2\pi(\lambda_1)^2 \\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}\ln L} &= 2\pi(\lambda_2)^2 \\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}\ln L} &= \lambda_3 - \pi\lambda_1\lambda_3 + 2\pi\lambda_2\lambda_3 + \pi\lambda_2\lambda_5 \\
\frac{\mathrm{d}\lambda_4}{\mathrm{d}\ln L} &= \frac{3}{2}\lambda_4 + \frac{3}{2}\pi\lambda_1\lambda_4 - 4\pi\lambda_4\lambda_5 \\
\frac{\mathrm{d}\lambda_5}{\mathrm{d}\ln L} &= \lambda_5 + 3\pi\lambda_1\lambda_5 + 3\pi\lambda_2\lambda_3.
\end{aligned} \tag{17}$$

The first and the second equations are decoupled from the others and can be integrated easily to yield

$$\lambda_1(L) = \frac{\lambda_1^{(0)}}{1 - 2\pi\lambda_1^{(0)}\ln(L/L_0)} \qquad \lambda_2(L) = \frac{\lambda_2^{(0)}}{1 - 2\pi\lambda_2^{(0)}\ln(L/L_0)}.$$

This implies that λ_2 is marginally relevant for the antiferromagnetic interchain coupling J_K, while it is marginally irrelevant for $J_\mathrm{K} < 0$. Therefore, the λ_2-interaction as well as the λ_1-interaction can be safely neglected in analysing the ferromagnetic regime $J_\mathrm{K} < 0$.

The remaining equations are integrated numerically and the result shows that the system always flows to the strong-coupling (i.e. large values of λ) region unless $J_\mathrm{K} = 0$ and $J' = 1$.

Thus we can conclude that the critical point of J_K for the homogeneous ($J' = 1$) case is given by

$$J_\mathrm{K}^\mathrm{c} = 0. \tag{18}$$

The critical exponent ν of the gap is easily read off from the third equation of (17). Neglecting the quadratic terms, we can readily solve it to yield $\lambda_3 = \lambda_3^{(0)}(L/L_0)$. This implies that the gap emerges both for $J_\mathrm{K} \to 0+$ and $J_\mathrm{K} \to 0-$ with the critical exponent

$$\nu = 1. \tag{19}$$

This implies that the transition at $J_\mathrm{K} = 0$ is of the second order. These results are in agreement with the recent numerical results [45]. Strictly speaking, however, the presence of marginal operators (λ_1 and λ_2) modifies the purely power-like behaviour in (19). Field theoretical argument similar to the one used in [36] predicts that the mass gap opens as

$$m(J_\mathrm{K}) \sim J_\mathrm{K}(\ln J_\mathrm{K})^{1/2}. \tag{20}$$

Recently, Fujimoto and Kawakami [46] applied a similar method to the Kondo lattice problem and obtained the result that the spin gap exhibits the essential singularity as $J_\mathrm{K} \to 0+$, which is consistent with the numerical calculation [47]. However, they neglected the relevant $\lambda_{3,5}$-interactions which lead to the power-like behaviour observed above. We do not know how to extract such an essentially singular behaviour in the presence of the relevant operators.

From the above analysis, we found that infinitesimally small interactions around the point ($J' = 1$, $J_\mathrm{K} = 0$) drive the system to the strong-coupling region. Combining this with the results obtained in section 2, we may expect the massive singlet phase to spread

over the antiferromagnetic region $J' > 0$, $J_K > 0$. In this phase, S- and T-spins form local singlets (pairwise or clusterwise) and the analysis using the continuum field theory breaks down.

However, the present analysis is insufficient for discussion of the ferromagnetic side, since a much more non-trivial phase diagram is expected from the known facts about $S = 1$ systems. In the strong-coupling regime, the original $SU(2)_S \times SU(2)_T$-symmetry is reduced down to the diagonal SU(2). Correspondingly, the $g \times \tilde{g}$-description based on the $SU(2)_S \times SU(2)_T$ symmetry is not a good starting point. Therefore, we adopt an alternative method based upon the diagonal SU(2) symmetry which is generated by the total spin $S_{tot} + T_{tot}$.

It is well known [41] that if $J_{L/R}$ and $\tilde{J}_{L/R}$ satisfy the level-1 Kac–Moody algebra then the diagonal currents $J^{diag}_{L/R} = J_{L/R} + \tilde{J}_{L/R}$ satisfy the level-2 one. There is a remarkable equivalence between two independent WZW models ($k = 1$) and a composite ($k = 2$ WZW) $\otimes$ (Ising) model, which is known as *the coset construction* [25, 26]. The above level-2 WZW model corresponds to the Kac–Moody algebra generated by the diagonal currents $J^{diag}_{L/R}$. Using this equivalence, we can rewrite our continuum model in the $g \times \tilde{g}$-description into the one in the ($k = 2$ WZW) $\times$ (Ising) description.

First, we begin by rewriting the interactions. It is well known [26] from the above quantum equivalence that bilinears of the characters of the level-1 Kac–Moody algebra are expressed in terms of the characters of the level-2 algebra and the $c = 1/2$ Virasoro algebra:

$$\begin{aligned} &\chi^{(1)}_{j=0}\chi^{(1)}_{j=0} = \chi^{(2)}_{j=0}\chi^{Vir}_{h=0} + \chi^{(2)}_{j=1/2}\chi^{Vir}_{h=1/2} \\ &\chi^{(1)}_{j=0}\chi^{(1)}_{j=1/2} = \chi^{(2)}_{j=1/2}\chi^{Vir}_{h=1/16} \\ &\chi^{(1)}_{j=1/2}\chi^{(1)}_{j=1/2} = \chi^{(2)}_{j=1/2}\chi^{Vir}_{h=0} + \chi^{(2)}_{j=0}\chi^{Vir}_{h=1/2}. \end{aligned} \tag{21}$$

These rules give a clue to how to decompose the interactions written via g and $\tilde{g}$ into those of the (Ising) $\otimes$ (WZW)-picture. All the necessary numerical coefficients are determined by requiring that the operator-product expansions should be consistent. After some algebra (see appendix B for a comment on the calculation), we obtain the desired results:

$$\begin{aligned} &\mathrm{Tr}\, g(z,\bar{z}) + \mathrm{Tr}\, \tilde{g}(z,\bar{z}) = \sqrt{2}\,\mathrm{Tr}\, h(z,\bar{z})\,\sigma(z,\bar{z}) \\ &\mathrm{Tr}\, g(z,\bar{z})\sigma + \mathrm{Tr}\, \tilde{g}(z,\bar{z})\sigma = \sqrt{2}\,\mathrm{Tr}\, [h(z,\bar{z})\sigma]\,\sigma(z,\bar{z}) \\ &\mathrm{Tr}\, g(z,\bar{z})\,\mathrm{Tr}\, \tilde{g}(z,\bar{z}) = \mathrm{Tr}\, \Phi_{j=1}(z,\bar{z}) + \varepsilon(z,\bar{z}) \\ &\mathrm{Tr}\, g(z,\bar{z})\sigma \cdot \mathrm{Tr}\, \tilde{g}(z,\bar{z})\sigma = \mathrm{Tr}\, \Phi_{j=1}(z,\bar{z}) - 3\varepsilon(z,\bar{z}) \end{aligned} \tag{22}$$

where the Ising operators σ (conformal weight $h = 1/16$) and ε ($h = 1/2$) denote the magnetization and the energy operator, respectively. For the level-2 WZW model, there are three primary fields; 1 (identity), h (the WZW fundamental field), and $\Phi_{j=1}$ (the spin-1 primary field expressed as a 3×3matrix). They appear in the right-hand side of (22), so that the transformation property under SU(2) of both sides may be consistent.

Using them, our model is rewritten as

$$\begin{aligned} S = S_{k=2\,WZW} + S_{Ising} + (\lambda_3 + \lambda_5)\int d^2x\ \mathrm{Tr}\, \Phi_{j=1}(z,\bar{z}) \\ +(-3\lambda_3 + \lambda_5)\int d^2x\ \varepsilon(z,\bar{z}) + \lambda_4 \int d^2x\ \sqrt{2}\,\mathrm{Tr}\, h(z,\bar{z})\,\sigma(z,\bar{z}). \end{aligned} \tag{23}$$

Note that we have extracted the $k = 2$ WZW model *non-perturbatively*. A simple perturbation using two $k = 1$ WZW models will never yield the $k = 2$ one.

For non-zero values of J_K, the second perturbation field ε drives the Ising sector off-critical. Therefore, we can ignore the massive Ising sector in investigating the low-energy behaviour only. After replacing the Ising fields by their expectation values, we obtain the effective theory of our problem:

$$S = S_{k=2\,\mathrm{WZW}} + \alpha \int \mathrm{d}^2x\ \mathrm{Tr}\,\Phi_{j=1}(z,\bar{z}) + \beta \int \mathrm{d}^2x\ \mathrm{Tr}\,h(z,\bar{z}). \tag{24}$$

Using equations (10) and (22), the 'fused' spin operator is expressed as

$$\mathcal{S}_i = \boldsymbol{S}_i + \boldsymbol{T}_i \approx \boldsymbol{J}_{\mathrm{L}}^{\mathrm{diag}} + \boldsymbol{J}_{\mathrm{R}}^{\mathrm{diag}} + \text{constant} \times (-1)^i\,\mathrm{Tr}\,[h\boldsymbol{\sigma}]\,. \tag{25}$$

In the next section, we analyse the model in detail.

4. Renormalization group and semiclassical treatment

In Section 3, we have found that when the interchain coupling J_K is turned on the Ising sector becomes massive. Thus the critical behaviour of the ladder model is governed by the remaining spin (SU(2)) sector, the level-2 SU(2) WZW model plus several interactions (24).

We calculate the β-function for the coupling constants α (the interchain coupling J_K) and β (bond alternation) following a similar method to that used in section 3. The resulting system of equations is given by

$$\begin{aligned} \frac{\mathrm{d}\alpha}{\mathrm{d}\ln L} &= \alpha - \pi\beta^2 \\ \frac{\mathrm{d}\beta}{\mathrm{d}\ln L} &= \frac{13}{8}\beta - 3\pi\alpha\beta. \end{aligned} \tag{26}$$

This system has three fixed points: (a) $(\alpha,\beta) = (0,0)$; (b) $\left(\frac{13}{12\pi}, \frac{1}{2\pi}\sqrt{\frac{13}{6}}\right)$; and (c) $\left(\frac{13}{12\pi}, \frac{-1}{2\pi}\sqrt{\frac{13}{6}}\right)$. The two points (b) and (c) are saddle points. The critical point $J_K^c = 0$ and the exponent $\nu = 1$ can be derived also from these equations, as should be the case. The result of a numerical integration is shown in figure 5. It clearly shows that two quite different kinds of behaviour occur. In the shaded portion, the present system flows to the strong-coupling region with $\beta = 0$; that is, the bond alternation is 'healed' in the low-energy limit. On the other hand, the flow starting from the other region except on the $\beta = 0$-line goes to the strongly dimerized ($|\beta| \gg 1$) phase. Therefore, we may expect that a transition occurs on the border of the two regions. Then a question arises: what is the ground state in these regions? To answer this, we adopt a semiclassical analysis first used by Affleck and Haldane [35] to predict the phase diagram of higher-S systems.

To begin with, we parametrize the fundamental field h of the level-2 WZW model in terms of an angle $\phi(x_1,x_2)$ ($0 \leqslant \phi < 2\pi$) and a unit vector $\widehat{\varphi}(x_1,x_2)$:

$$h(x_1,x_2) = \exp\left(\frac{\mathrm{i}}{2}\phi(x_1,x_2)\boldsymbol{\sigma}\cdot\widehat{\varphi}(x_1,x_2)\right). \tag{27}$$

Note that the discrete symmetry $h \to -h$ corresponds to $\phi \to -\phi$, $\widehat{\varphi} \to -\widehat{\varphi}$ in this parametrization. Then, we put this into the action of the $k = 2$ WZW model plus interactions†

$$\alpha \int \mathrm{d}^2x\ (\mathrm{Tr}\,h)^2 + \beta \int \mathrm{d}^2x\ \mathrm{Tr}\,h\,.$$

† Taking the operator-product expansion (OPE), we can see that $(\mathrm{Tr}\,h)^2$ actually corresponds to the spin-1 primary field. However, we cannot accept this equality literally, because some multiplicative renormalization is necessary in obtaining the spin-1 field from OPE of the h-fields.

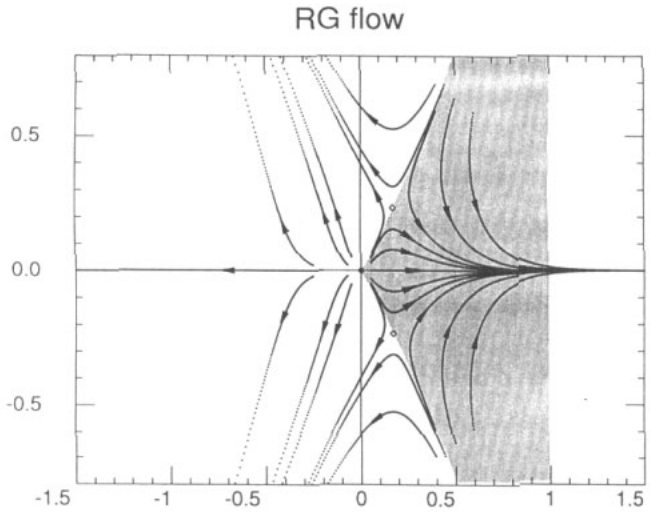


Figure 5. The renormalization group flow obtained by integrating (26). In the shaded portion, the system is renormalized onto the $\beta = 0$ line, i.e. the bond alternation is healed. The three fixed points are plotted by a symbol '$\diamond$'.

After some algebra, we obtain

$$S[\phi, \widehat{\varphi}] = \frac{1}{8\pi}\int \mathrm{d}^2x \left[(\partial_\mu \phi)^2 + 2(1-\cos\phi)(\partial_\mu \widehat{\varphi})^2\right] + \frac{1}{8\pi} 2(\phi - \sin\phi)\int \mathrm{d}^2x\ \epsilon_{\mu\nu}\widehat{\varphi}\cdot(\partial_\mu\widehat{\varphi}\times\partial_\nu\widehat{\varphi}) + \int \mathrm{d}^2x \left[4\alpha\cos^2\left(\frac{\phi}{2}\right) + 2\beta\cos\left(\frac{\phi}{2}\right)\right] \tag{28}$$

where the first, second, and third lines come from the kinetic, the Wess–Zumino, and the potential terms, respectively. Following the standard procedure, we look for a uniform solution of ϕ which minimizes the potential term. As a result, we obtain the effective action governing the dynamics of the remaining $\widehat{\varphi}$-field:

$$S_{\mathrm{eff}}[\widehat{\varphi}] = \frac{1}{2}\frac{(1-\cos\langle\phi\rangle)}{2\pi}\int \mathrm{d}^2x\ (\partial_\mu\widehat{\varphi})^2 + \frac{1}{8\pi} 2(\langle\phi\rangle - \sin\langle\phi\rangle)\int \mathrm{d}^2x\ \epsilon_{\mu\nu}\widehat{\varphi}\cdot\left(\partial_\mu\widehat{\varphi}\times\partial_\nu\widehat{\varphi}\right) \tag{29}$$

up to an unimportant constant term. It is important to note that this is simply the action of the O(3) nonlinear sigma model with the topological angle

$$\Theta_{\mathrm{top}} = 2(\langle\phi\rangle - \sin\langle\phi\rangle).$$

The result is summarized in figure 6.

In the region indicated by 'dimerized', the solution is given by $\phi = 0$ or 2π and the $\widehat{\varphi}$-field becomes infinitely massive. The fluctuation around $\phi_{\mathrm{min}} = \langle\phi\rangle$ is massive as well. On the other hand, we obtain $\langle\phi\rangle = \pi$ on the half-line $\alpha > 0$ and $\beta = 0$. In this case, S_{eff} becomes the action of the $\Theta_{\mathrm{top}} = 0$ nonlinear sigma model, which dynamically generates a mass in the low-energy limit. It is worth mentioning that this model is believed to be an effective model which describes the so-called Haldane systems [10, 48].

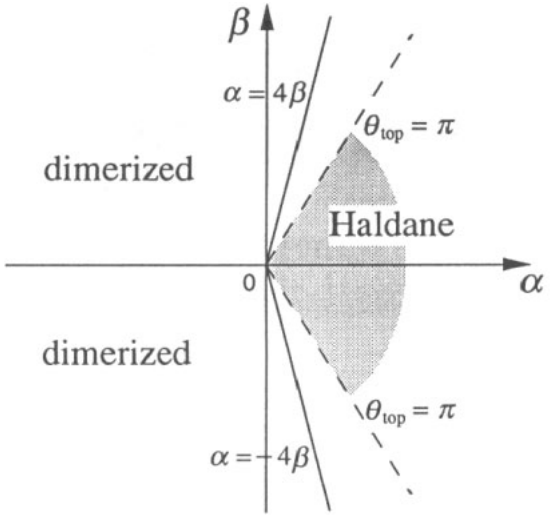


Figure 6. The result of the semiclassical analysis. On the dashed lines, the system is described by the O(3) nonlinear sigma model with $\Theta_{\text{top}} = \pi$. This corresponds to the Haldane–dimer transition.

For other values of α and β satisfying the inequality $-4\alpha < \beta < 4\alpha$, the potential is minimal when $\langle\phi\rangle = 2\cos^{-1}(-\beta/4\alpha)$. We have estimated the value of β/α for which the topological angle becomes π. Our result is given by $\beta/\alpha = \pm 1.62$. Therefore, we have the O(3) nonlinear sigma model with $\Theta_{\text{top}} = \pi$ as an effective action of the $\widehat{\varphi}$ on the lines $\beta = \pm 1.62\alpha$. In the region between these lines, the topological angle Θ_{top} takes intermediate values between 0 and π. However, the renormalization group result suggests that in some region around $\beta = 0$ the system renormalizes onto the $\beta = 0$ line, on which Θ_{top} is given by 0.

Therefore, we may expect that (i) there are two transition lines on which the system is critical and that (ii) the region between them corresponds to the Haldane phase characterized by the $\Theta_{\text{top}} = 0$ sigma model. On the transition lines, the effective theory is given by $\Theta_{\text{top}} = \pi$ model, which is believed to be equivalent to the level-1 SU(2) WZW model [35] [39]. In a recent work [32], it was shown that the critical point of the $S = 1$ Haldane–dimer transition belongs to the universality class of the level-1 SU(2) WZW model. Thus, combining this with the result of the perturbation expansion presented in section 2, we expect that the transition lines found near the point $(J', J_{\text{K}}) = (1, 0)$ are connected to the $S = 1$ transition point. Along the line, the string order parameter defined on a ladder [14] will vanish with the exponent 1/6 [32] and the spin correlation functions decay as $1/r$.

To conclude this section, a remark is in order about the so-called *composite-spin model* [49, 50]. This model is a variation of the present ladder model where the rung interactions are replaced by the *cross* interactions. In their study of the model, Sólyom and Timonen [50] found that the system exhibits the same criticality as the integrable Takhtajan–Babujian model [51, 52] for some appropriate choice of the couplings. According to Affleck and Haldane [34, 35], this solvable model is described by the level-2 SU(2) WZW model in the low-energy limit. From our point of view, their choice of the coupling constants corresponds to the fine-tuning of the relevant couplings α and β.

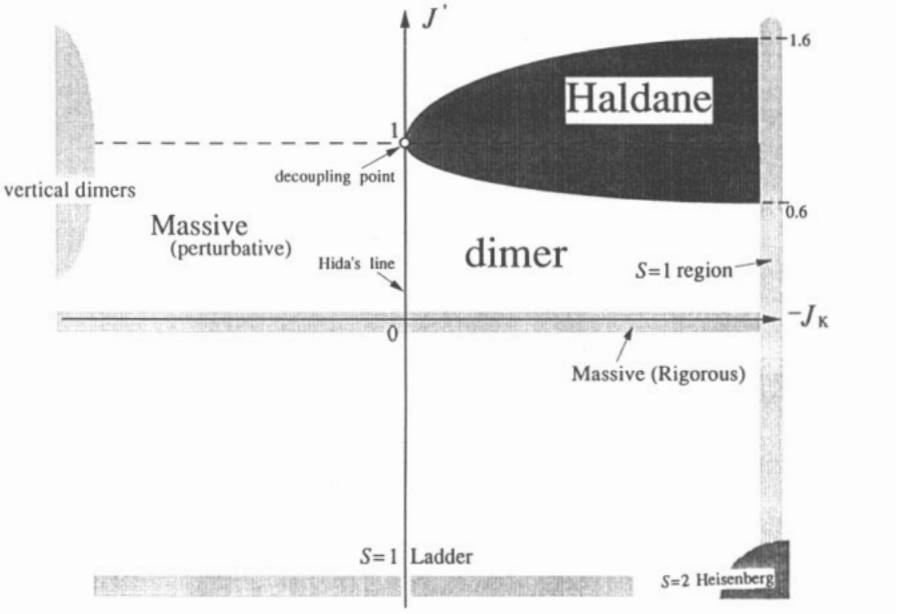


Figure 7. A conjectured phase diagram. The so-called Haldane phase originates at the decoupling point ($J_K = 0, J' = 1$) and is expected to be connected to the $S = 1$ limit. The universality class of the Haldane–dimer transition is the same as that of the $S = 1/2$ Heisenberg model.

5. Summary and discussions

In the preceding sections, we have investigated the ground-state phase diagram of the alternating ladder model (1). The results are summarized in a phase diagram (figure 7).

From the perturbation calculations and the bosonization analysis performed in sections 2, 3, and 4, it is suggested that the short-ranged singlet phase spreads over all the antiferromagnetic region ($J_K > 0$). Both numerical results [4, 31] and the results of our perturbation expansions imply that the crossover to the strong-coupling ($J_K \gg 1$) region is rather fast. Therefore, we expect that the vertical dimer or singlet-A (see section 2) configuration dominates even for relatively small values of J_K. Within a simple variational calculation, the singlet-A configuration always has an energy lower than the vertical-dimer one.

In order to explore the ferromagnetic region ($J_K < 0$), we have bosonized the system in terms of the level-2 WZW model and the Ising model. This kind of separation of degrees of freedom does not occur in the ordinary perturbative calculation based on two WZW models. That is, the appearance of the $k = 2$ model is non-perturbative.

The Ising sector becomes massive for non-zero values of J_K. The renormalization group flow of the remaining theory shows two types of limiting behaviour; one flow converges to the system with no bond alternation and the other goes to a trivial strong-coupling fixed point, where the system becomes strongly dimerized and the field theoretical description breaks down. Between these phases, a certain type of transition will occur.

Using the semiclassical analysis, we have argued that there exist second-order transition lines which correspond to the level-1 WZW model, which belongs to the same universality class as the $S = 1/2$ Heisenberg chain. On the ferromagnetic side of them, a translationally invariant massive phase is realized and is connected to the Haldane phase of the $S = 1$

Heisenberg chain. In other words, the Haldane phase of the $S = 1$ chain 'originates' in the $S = 1/2$ Heisenberg point ($J' = 1, J_K = 0$). Note that it is the only point where the translational invariance is not broken in the $S = 1/2$ chain. It is not easy to discuss the string order in our treatment using the WZW model. However, a variational analysis [53] based on a ladder analogue of the VBS state suggests that the string order develops in this region (see figure 3).

On the other hand, our semiclassical analysis tells us that a kind of dimerized phase, which breaks the translational invariance: $h \to -h$, occurs outside the above region, as is expected from the perturbation expansion or the renormalization group analysis.

Finally, we comment on the extension of our method to spin ladders with more than three legs. For simplicity, we consider the three-leg case below. We have SU(2)×SU(2)×SU(2) symmetry at the decoupling point $J_K = 0$. Just as in the present two-leg case, we can decompose the product of three $SU(2)_1$ (the suffix '1' denotes the level) into $SU(2)_3$ and a certain coset CFT whose central charge equals 6/5. In section 3, this residual coset CFT was given by the well known Ising model. If we assume that the residual sector becomes massive by introducing J_K as in the two-leg case, we are left with the level-3 SU(2) WZW model with several interactions. A similar analysis to that in section 4 may be applied. In this case, a classical solution $\langle\phi\rangle = \pi$ corresponds to the $\Theta_{top} = \pi$ sigma model (see (29)) which is expected to be massless [54, 39]. Note that the solution $\langle\phi\rangle = \pi$ does not break the translational invariance by one site. Hence we expect that the gapless ground state is realized in the homogeneous chain (i.e. with no bond alternation). This is consistent with the observation of Timonen *et al* [55] and White *et al* [6] as well as the Haldane conjecture itself.

Acknowledgments

The author (KT) thanks Professor A Kato for helpful comments. He also benefited from conversation with Y Nishiyama and M Onoda.

Appendix A. The second-order result of the weak-coupling expansion

In section 2, we gave a result up to first order in J'. We improve it by calculating the second-order terms.

As in section 2, the calculation is rather straightforward. However, we have to take into account all excited states of a cell in this case and this makes our calculation complex and tedious. After a long computation, we arrive at the final result

$$\begin{aligned}\omega^{(2)}(k) &= \sum_{n\neq\text{triplet}-A} \frac{|\langle n|\widehat{V}|\{k\}\rangle|^2}{E_{\text{triplet}-A} - E_n} - \sum_n \frac{|\langle n|\widehat{V}|\text{G.S.}\rangle|^2}{E_{\text{G.S.}} - E_n} \\ &= \frac{(J')^2}{8(1+\alpha+\alpha^2)} \left\{ \frac{-5}{3}\frac{1}{\alpha+2} + \frac{1}{2(\alpha - J_K + 1)} \left(\frac{3(\alpha+1)^4}{1+\alpha+\alpha^2} - \frac{1}{1+\beta+\beta^2} \right) \right. \\ &\quad - \frac{\alpha^2(1+\cos k)}{\alpha - 2J_K + 3} - \frac{2\alpha(1+\alpha)(1-\cos k)+1}{\alpha - 2J_K + 2} \cdot \\ &\quad - \left(1 + \cos k - \frac{3 - \cos 2k}{2(1+\alpha+\alpha^2)} \right) \frac{1}{\alpha+1} \\ &\quad \left. - \frac{(1+\alpha)^2(1+\cos k)}{\alpha - 2J_K + 1} + \frac{3}{2}\frac{\alpha^4}{1+\alpha+\alpha^2}\frac{1}{\alpha - J_K + 2} \right.\end{aligned}$$

$$+\frac{6\alpha^2(1+\alpha)^2}{1+\alpha+\alpha^2}\frac{1}{2\alpha-2J_K+3}\Bigg\}. \tag{A1}$$

As is described in section 2, the ground state $|G.S.\rangle$ is given by the state *singlet A* and $|\{k\}\rangle$ denotes a plane-wave state of *triplet-A* with a momentum k. The parameter $\alpha(J_K)$ is given by

$$\alpha(J_K) = J_K - 1 + \sqrt{J_K^2 - J_K + 1}.$$

For $J_K \to -\infty$, $\omega^{(2)}(k)$ reduces to

$$\omega^{(2)}(k) = (J')^2\left(\frac{4}{27} - \frac{1}{3}\cos k - \frac{2}{9}\cos 2k\right). \tag{A2}$$

Combining this with the first-order result, we obtain the dispersion relation for the elementary excitation over the $S = 1$ dimerized chain:

$$\omega(k) = \frac{1}{2} - \frac{2}{3}J'\cos k + (J')^2\left(\frac{4}{27} - \frac{1}{3}\cos k - \frac{2}{9}\cos 2k\right). \tag{A3}$$

The gap appears at $k = 0$ and it vanishes for $J' = 0.557$ to this order. This is to be compared with the known critical values of the Haldane–dimer transition $J' \approx 0.6$ [56, 32].

Appendix B. A remark on the calculations of sections 3 and 4

In this appendix, we give a brief comment on the calculation performed in sections 3 and 4. In these sections, we frequently used the operator-product expansion (OPE) [57]. For example, the 1-loop β-function is simply given by [42]

$$\frac{d\lambda_k}{d\ln L} = (2 - x_k)\lambda_k - \pi\sum_{i,j} C_{ij}^k \lambda_i \lambda_j \tag{B1}$$

where the quantity x_k stands for the scaling dimension of the perturbing field ϕ_k and the numerical coefficients C_{ij}^k are defined by the following OPE:

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) \sim \sum_k \frac{1}{|z-w|^{x_i+x_j-x_k}} C_{ij}^k \phi_k(w,\bar{w}).$$

For example, the SU(2) currents in section 3 obey the following short-distance expansion:

$$J_L^\alpha(z)J_L^\beta(w) = \frac{k/2}{(z-w)^2}\delta_{\alpha\beta} + \frac{i\epsilon_{\alpha\beta\gamma}}{z-w}J_L^\gamma(w) \tag{B2}$$

which is called the (level-k) Kac–Moody algebra.

The OPE coefficients $\{C_{ij}^k\}$ are related to the three-point function $\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle$ of the primary fields. The latter is worked out in detail in [40] for the SU(2) WZW model. Once we have the numerical coefficients $C_{ijk}^{3\text{-pt}}$ of the three-point functions, we readily obtain the OPE coefficients as

$$C^{(j_3,m_3,\bar{m}_3)}_{(j_1,m_1,\bar{m}_1)(j_2,m_2,\bar{m}_2)} = (-1)^{j_3+m_1+m_2}(-1)^{j_3+\bar{m}_1+\bar{m}_2}C^{3\text{-pt}}_{(j_1,m_1,\bar{m}_1)(j_2,m_2,\bar{m}_2)(j_3,-m_3,-\bar{m}_3)} \tag{B3}$$

where the parameters j_1, j_2, and j_3 label the spins of the primary fields. The numerical factors $\{C^{3\text{-pt}}\}$ are obtained by obtaining the necessary derivatives in equation (3.14) of [40]. Other necessary coefficients can be found in [58].

Note added in proof. After completion of this work, the author (KT) received a preprint from Hida [59]. He estimated a possible logarithmic correction (see equation (20)) to obtain the power of $\ln J_K$ as 0.4 ± 0.07. This seems consistent with our result.

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